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Power laws in cities population, financial markets and internet sites (scaling in systems with a variable number of components)

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Abstract

We study a few dynamical systems composed of many components whose sizes evolve according to multiplicative stochastic rules. We compare them with respect to the emergence of power laws in the size distribution of their components. We show that the details specifying and enforcing the smallest size of the components are crucial as well as the rules for creating new components. In particular, a growing system with a fixed number of components and a fixed smallest component size does not converge to a power law. We present a new model with variable number of components that converges to a power law for a very wide range of parameters. In a very large subset of this range, one obtains for the exponent α the special value 1 specific for the city populations distribution. We discuss the conditions in which α can take different values. In the case of the stock market, the distribution of the investors' wealth is related to the ratio between the new capital invested in stock and the rate of increase of the stock index.

1. Power laws

Power laws have been discovered more than a hundred years ago by Vilifredo Pareto [1]. Pareto discovered that the relative number of individuals $Q(>w)$ with an annual income larger than a certain value w is proportional to a power of w:

$$Q(>w) \sim w^{-\alpha}\,. \tag{1}$$

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Other ways to express the same relation are:

- in terms of the probability $P(w)\,\mathrm{d}w$ for a person to have an income between w and $\mathrm{d}w$

$$P(w)\,\mathrm{d}w \sim w^{-1-\alpha}\,\mathrm{d}w\,. \tag{2}$$

- by expressing the income $W(n)$ of the various individuals in descending order: $W(1)$ being the income of the person with the highest income, $W(2)$ the income of the person with the second highest income and so on

$$W(n) \sim n^{-1/\alpha}\,. \tag{3}$$

The plot of $W(n)$ on a double logarithmic scale is called a "Zipf plot" [2] and leads in the case of a power law (Eq. (3)) to a straight line with slope $-1/\alpha$.

In the meantime, similar "power laws" were discovered in a wide range of other fields: individual wealth distribution [3], words frequency [2], etc.

One of man-made structures, known to exhibit power-law probability distribution is the population in cities all over the world. There have been several observations in the past, which have shown that this probability distribution obeys the general rule [2,4,5]

$$N(w) \sim w^{-1-\alpha}\,, \tag{4}$$

where $N(w)$ is the number of cities with population w and α is the power law parameter, which was measured to be 1.

In the present paper we start from some recently proposed models, which predict the empiric observation described by Eq. (4), explain and demonstrate their limitations and shortcomings and propose a more realistic and successful model.

2. Previous models and results

Recently [6], Gabaix has suggested that the behavior described by Eq. (4) can be explained by a modified version of the following model of Levy and Solomon [7].

Assume an auto-catalytic process [8] characterized by the rule that the growth of each individual entity is proportional to its present size. More precisely, assume a constant number N of cities i whose populations change from year to year according to the rule

$$w_i(t+1) = \lambda_i(t) w_i(t)\,, \tag{5}$$

where $w_i(t)$ is the population of the ith city at time t, and $\lambda_i(t)$ is a random number, extracted from a distribution $\pi(\lambda)$ (independent of i and w_i). A realistic distribution $\pi(\lambda)$ must insure the experimentally observed yearly increase in overall world population (about 1.02) and must have a standard deviation large enough to enable occasionally, values which are smaller than 1 ($\sigma \approx 0.05$). In such conditions, obviously both the total population

$$w_{tot}(t) = \sum_i w_i(t) \tag{6}$$

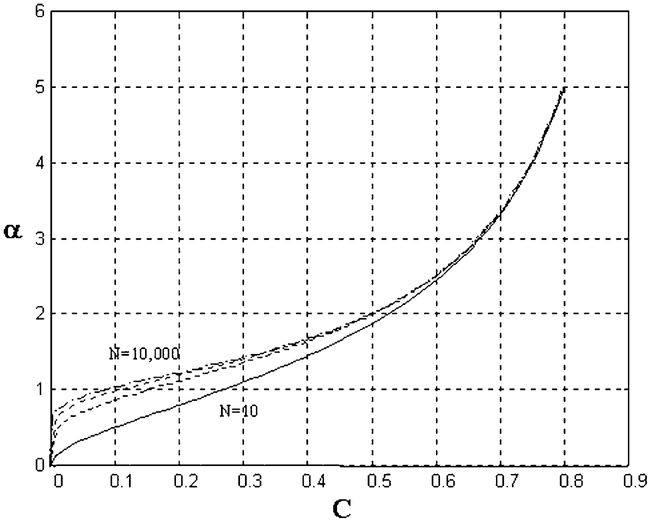

Fig. 1. α as a function of $c = w_{\min}/\langle w \rangle$ for various N values ($N = 10, 100, 1000$ and $10\,000$). One sees that for any finite N one obtains $\alpha(0) = 0$. Yet the range of c's with $\alpha(c) < 1$ shrinks as $1/\ln N$ when $N \to \infty$. One sees from the figure that for $c = 0$ and finite N one always gets $\alpha = 0$ but for $N = \infty$, the accumulation point for $c \to 0$ is $\alpha = 1$. This implies that there is a non-uniform limit of α in terms of c and N.

and the average city population

$$\langle w \rangle(t) = w_{tot}(t)/N \tag{7}$$

increase for large times indefinitely.

If one studies the above generic multiplicative process of Eq. (5) with no additional constraints, one gets closer and closer to a power law described by Eq. (2) with exponent $\alpha = 0$. It was shown however that if no $w_i(t)$ is allowed to become less than a minimal value $w_{\min}$, then the result can be dramatically different. In the model introduced by Levy and Solomon [7], $w_{\min}$ was a fixed (in time) fraction c of the current average $\langle w \rangle(t)$

$$w_{\min}(t) = c\langle w \rangle(t)\,. \tag{8}$$

Operationally, each time that Eq. (5) returned a value $w_i(t+1)$ smaller than $w_{\min}(t)$, the actual value of $w_i(t+1)$ was updated to

$$w_i(t+1) = c\langle w \rangle(t)\,. \tag{9}$$

Obviously, Eq. (8) implies that if c is time independent, $w_{\min}$ varies in time.

This dynamics was shown theoretically and numerically to lead to a very a stable power law with $\alpha > 0$ given by the implicit equation [8]

$$N = [(1 - (N/c)^{\alpha})/\alpha]/[(1 - (N/c)^{\alpha-1})/(\alpha - 1)]\,. \tag{10}$$

The actual solutions $\alpha(c, N)$ of this equation for various N are plotted in Fig. 1.

More quantitatively, in the limit

$$N \gg e^{1/c} \equiv e^{\langle w \rangle / w_{\min}} \tag{11}$$

Eq. (10) reduces to the relation [7]

$$\alpha \approx \frac{1}{1-c} \equiv \frac{1}{1-w_{\min}/\langle w \rangle}\,, \tag{12}$$

which implies, in particular, $\alpha > 1$ for all $w_{\min}$ values and $\alpha \approx 1$ for small $w_{\min}/\langle w \rangle \equiv c \ll 1$ (*as long as the inequality Eq.* (*11*) *holds*). Both the exact Eq. (10) and the range of validity (Eq. (11)) of the approximation Eq. (12) were confirmed numerically in [8].

Gabaix [6], unaware of the condition Eq. (11) for the validity of the Eq. (12) concluded that for time fixed $w_{\min}$ and N and for $\pi(\lambda)$ distributions that insure the continuum increase of $\langle w \rangle(t)$ one obtains automatically $w_{\min}/\langle w \rangle \equiv c \to 0$ and therefore $\alpha \approx 1$ following Eq. (12).

This conclusion in [6] is mistaken because taking $w_{\min}$ and N fixed in time implies that w_{tot}, $\langle w \rangle$, and c are time dependent. In particular, as $\langle w \rangle(t)$ increases c decreases for fixed $w_{\min}$ and therefore, for fixed N, condition (11) is eventually violated for all times larger than a certain (rather small) value of t.

In the present paper we verify numerically the arguments and show that as opposed to the case studied by Levy and Solomon [7] in which c is fixed and $w_{\min}$ is time varying, the modification introduced by Gabaix [6] in which $w_{\min}$ is fixed (and therefore $c \to 0$ as $t \to \infty$) does not converge to a power law. In fact, even if one tries to fit locally a power law by measuring the tangent on a double logarithmic plot of $P(w)$, the measured slope varies continuously and corresponds after some time to $\alpha \ll 1$.

Moreover, we show that the statement in [6] that a time variation of the total number of cities N is irrelevant for the behavior of the system is wrong. On the contrary, we show that if the number of cities N increases proportionally to the increase in the total population in the system

$$N(t+1) - N(t) = K(w_{tot}(t+1) - w_{tot}(t))\,, \tag{13}$$

a power law is recovered with an exponent $\alpha \sim 1$ for a wide range of reasonable population growth distributions $\pi(\lambda)$ and city proliferation parameters K. Such a model has the advantage that

(1) The smallest allowed city population $w_{\min}$ can be kept to realistically small values (O(1)) during the entire evolution of the system.

(2) The number of cities N is increasing realistically in time.

(3) One can afford now to allow the cities which become smaller than a minimal value (of $w_{\min} \sim O(1)$) to disappear without the danger of the system to be left eventually without cities at all.

This modification has also applications in econo-physics [9–11] since it models stock markets with an increasing number of investors and with investors that can drop from the market.

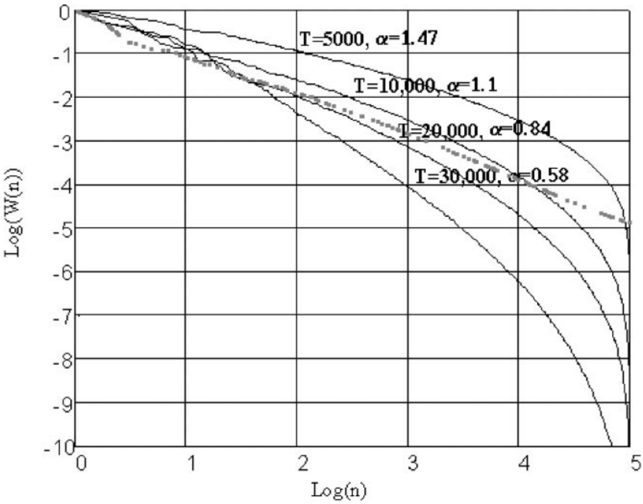

Fig. 2. The solid lines are the "Zipf plots" as obtained from the model [6] with fixed $w_{\min} = 1$. One sees that they are not close to straight lines and that their slope $1/\alpha$ gets steeper and steeper in time to values definitely larger than 1. We superpose for comparison (the dashed line) the results for the model [7] with fixed $c = 0.1$. One sees that on the double logarithmic graph this model converges to a precise straight line with $\alpha = 1.04$.

3. The simulations of the Gabaix model

As mentioned above, in the Gabaix version [6] of the Levy and Solomon model [7], $w_{\min}$ was a time-independent constant unrelated to $\langle w \rangle(t)$. More precisely, the population of no city is allowed to become smaller than 1. Operationally, if by the application of the Eq. (5) at a certain instant, the population of a city decreases below 1, then the prescription in [6] is to "force it back" to 1 "by hand".

A typical numerical result, based on this prescription is shown in Fig. 2.

The various curves are plots of $W(n)$ at various time instances (the curves are steeper for later times). It can be seen that for N and $w_{\min}$ fixed, the system does not converge to a power law. Even if one insists in measuring the slope of the Zipf plot and calling it $-1/\alpha$ one gets values very different (larger and larger) from 1 (in absolute values). This is easily understood if one notices that for exponentially increasing $\langle w \rangle \sim O(1.02^t)$ and fixed N and $w_{\min}$, the condition $N \gg e^{\langle w \rangle / w_{\min}}$, in Eq. (11) stops from being fulfilled long before $P(w)$ has the chance to relax to a power-law shape.

For comparison we reproduce on the same graph the very stable power law $P(w)$ with $\alpha \sim 1$ obtained by coupling through Eq. (8) the lower bound $w_{\min}$ to the average $\langle w \rangle$ [7] (the graph is obtained for fixed $c = 0.1$ and $N = 100\,000$).

4. The definition, properties and simulation of the models with city proliferation

While the model with fixed $w_{\min}$ and fixed N [6] turns out to be invalid for producing power laws, one is interested to relax the assumptions used in the generic power-law generating model of [7] and to adapt them to the particularities of the population and cities dynamics.

In particular, we wish to endow the model with the following properties:

(1) *The existence during the entire history (and independent on the values of the total earth population) of small villages of the order of a few inhabitants.* i.e., relaxing the requirement that the smallest allowed city size $w_{\min}$ grows (Eq. (8)) with the average city size.

(2) *The disappearance of cities and villages that shrink below the minimal size* $w_{\min}$. As opposed to the economics examples, which were the focus of [7], for cities there is not such a thing as "subsidizing" a city "poor" in citizens. Therefore, it is not natural to assume that cities which shrink below the minimal size (e.g. below $w_{\min}=1$) will be restored to that value. On the contrary it is natural to assume that they will disappear.

(3) *The establishment of new cities.* It is reasonable to assume that as the population increases, some of the new citizens will wish to establish new cities. One may assume that in each given large sample of new citizens, there is a certain (small) fraction likely to establish new cities (which will start with the minimal size).

Each of the modifications 1–3 proposed above poses problems in preserving the power law, or the dynamical control of the system, or the stability of the exponent α. We will see that when the modifications 1–3 are implemented simultaneously, they find naturally their own solution.

As we have seen in the analysis above, the main problem with introducing item (1) is that as $w_{tot}(t)$ increases, so does $\langle w\rangle(t)=w_{tot}(t)/N$ and therefore $c=w_{\min}/\langle w\rangle(t)$ becomes smaller and smaller until the inequality Eq. (11) $e^{1/c}\ll N$ is violated and Eq. (10) does not reduce to Eq. (12). One sees now from Eq. (11) that item (3) might provide a solution to this problem.

If we could arrange that N increases in time proportionally to the increase in $w_{tot}(t)$, then $\langle w\rangle(t)$ would remain constant and so would c.

In such a situation, the value $\alpha=1/(1-c)$ will stay constant and the inequality $e^{1/c}\ll N$ insuring its validity will only improve with time (since N will increase exponentially with time).

Point (2) will loose its harm: the disappearance of small villages will not imply the depletion of N since it is now (over-) compensated by the exponential increase of N due to the emergence of new cities.

Let us now formulate the new model:

- The individual city populations vary as determined by Eq. (5) except if $w_i(t+1)<w_{\min}(=O(1))$ in which case the city *disappears*.
- At each time tick, there will appear a number

$$\Delta N=K(w_{tot}(t+1)-w_{tot}(t)) \tag{14}$$

of new cities of minimal size $w_{\min}$.

The relevant range of the real parameter K will be discussed below.

This model achieves exactly what we wanted: following the dynamics of N (Eq. (14)) the average $\langle w \rangle$ becomes constant

$$\langle w \rangle \to \frac{\Delta W}{\Delta N} = 1/K \tag{15}$$

and consequently so does

$$c = w_{\min}/\langle w \rangle \, . \tag{16}$$

This in turn implies that following Eq. (12), and due to the fact that Eq. (11) holds, α becomes a constant too. We therefore expect this model to converge robustly to a power law. The actual value of α can be calculated by estimating the time constant value of $\langle w \rangle$ from the present day data but it is not important to find $\langle w \rangle$ with great precision because any $c = w_{\min}/\langle w \rangle$ value in the range required by Eq. (11) $1/\ln N < c \ll 1$ will give according Eq. (12) an exponent $\alpha \sim 1$. The actual value of $\langle w \rangle$ can then be recovered from the experimental data using Eq. (3) with $\alpha \sim 1$ and recalling that the current total earth population is $O(10^{10})$ and the population of the largest city is currently $O(10^7)$:

$$\langle w \rangle = w_{tot}/N \approx w_{tot} w_{\min}/W(1) \approx 10^{10}/10^7 = 10^3 \, . \tag{17}$$

We also assumed in Eq. (17) that the "smallest city" population is O(1). The value of $\langle w \rangle$ fixes the realistic range to be used for K in the actual runs

$$K \sim 1/\langle w \rangle \sim 0.001 \, . \tag{18}$$

Again, the exact value is not important as long as $1/\ln N < c \ll 1$ where the new formula for c is (cf. Eqs. (16) and (18)):

$$c = K w_{\min} = \frac{w_{\min} \Delta N}{\Delta W} \, . \tag{19}$$

As described above, the mean value of λ, and its standard deviation are to be taken as to reflect the typical yearly population growth. We take typically in our runs $\langle \lambda \rangle \sim 1.02$ and its standard deviation 0.05 but the exact values have no substantial influence on the result of population distribution.

The initial conditions of this model can be chosen without much restriction. However, we believe that it is most appropriate to begin with a few people that do not belong to any city. This pool will grow according to Eq. (5) until its population will be large enough that $K \Delta w_{tot} \sim 1$ and cities will start emerging according to Eq. (14).

A typical result of our model is shown in Fig. 3 that presents the Zipf graph for $K = 0.001$. In contrast to the model in [6], for which we could not observe any convergence even after 30 000 years, the present model converges to a power law within a few hundred years. In fact we observed a very stable power law with $\alpha \sim 1$, for a wide range of K's involving a few orders of magnitude. Of course for extremely small values of the parameter K's the approximation Eq. (11) may break and values of α smaller than 1 are possible cf. Eq. (12).

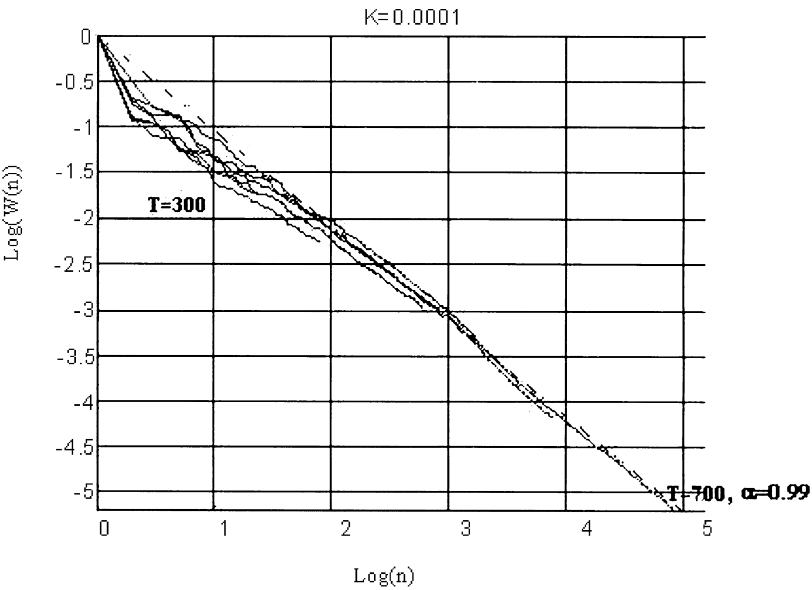

Fig. 3. Results for the model proposed in Eq. (14) for $K = 0.001$. The Zipf plot converges to a power law with stable $\alpha = 1$, after a few hundred steps. A dashed line of $\alpha = 1$ is drawn for comparison.

5. Applications of the model to financial, internet links and other systems

The models described above can be considered as mean-field approximations of the spatially distributed population models studied in [12,13]. In those models, the citizens can “diffuse” on a d-dimensional lattice and multiply when they meet appropriate conditions. The local conditions are represented by a time-dependent distribution of resources. It was shown that in such models, the population self-organizes in localized spatio-temporal adaptive patches associated with the regions rich in resources. In the “mean field limit” when the agents can diffuse between locations situated at arbitrary distance, one recovers the power-law distribution (Eqs. (1)–(4)) which in turn when represented in the “real space” implies a fractal spatial distribution of the population [14–19].

These models are not restricted to city population, the space may represent the genetic space in which case the power law applies to the sizes of species or the space of investing opportunities in which case the power law applies to investor herds, to companies capitalization, or to the individual wealth [20]. If one interprets $w_i(t)$ as the wealth invested in stock by an investor i then, the value $\alpha \sim 1.4$ specific to the investor wealth distribution can be related to the ratio Eq. (19) between the rate of increase in the total stock market worth and the rate of new investments. This quantity, is basically the global market impact factor

$$F = (\mathit{total\ market\ worth\ variation})/(\mathit{new\ investments}) = \frac{\Delta W}{w_{\min}\Delta N} = 1/c\,. \quad (20)$$

One sees that an $\alpha \sim 1.4$ implies a global impact factor of

$$F = 1/c = 1/(1 - 1/\alpha) \sim 3.5\,. \tag{21}$$

During the recent (and still on-going at the time of writing this article) Nasdaq bubble, the factor F seems to become larger and has indeed brought to a decrease in the absolute value of α, i.e., to an increase in the relative proportion of the total wealth possessed by the wealthiest.

Recently, power laws were found for the number of links pointing to internet sites (and for the number of visits therein) [21].

If one assumes that

- the probability for a internet site to get a new link is proportional to the present number of links $w_i(t)$ pointing to this site (Eq. (5)) and that
- the total number of new added links $W_{tot}(t+1) - W_{tot}(t)$ is proportional (by a factor $L = 1/K$) to the number of new added sites ΔN (Eq. (14)).

then, our model and analysis above predict a power-law distribution of the number of links pointing to various sites $P(w)$ (Eq. (4)) which according to Eq. (10)–(12), has for $N \gg e^L$ an exponent

$$\alpha \approx 1 + 1/L$$

and for $N \ll e^L$:

$$\alpha \approx 1 - \ln L / \ln N\,.$$

Note that the results in this paper are independent on time variations in the average $\langle \lambda_i(t) \rangle$ in Eq. (5), therefore as shown in [22], the scaling law Eq. (1) with stable exponent α Eq. (10)–(12) holds for arbitrary variations of $w_{tot}(t)$ including population depletion caused by wars, famine or epidemies and baby-booms induced by improved health-care and prosperity.

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